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Twistor spinors and their zeroes *

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Abstract

A generalization of the Einstein condition for Killing spinors is given for twistor spinors on Riemannian manifolds. We study the zeroes of twistor spinors on manifolds with parallel Ricci-tensor, in particular on Einstein manifolds. Furthermore, we consider the conformal deformation of the metric defined by a twistor spinor and find a sufficient condition for the completeness of this metric on the complement of the zero set. Examples are constructed for which the situation is realized. Finally, we obtain results concerning the conformal vector field defined by a twistor spinor.

Key words: twistors, spinors, Riemannian geometry,
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Introduction

P. Penrose introduced twistor spinors on a spin manifold in general relativity (see for instance [27]). In [1] M.F. Atiyah, N.J. Hitchin and J.M. Singer follow the idea of P. Penrose. Using the twistor equation they prove an integrability condition for the complex structure on the twistor space of an oriented four-dimensional Riemannian manifold. Further, it is remarkable that the twistor spinors correspond to parallel sections in a certain bundle (see for instance [6] or [12]). In this paper we study n-dimensional Riemannian spin manifolds (M^n, g), where $n \geq 3$, admitting non-trivial twistor spinors. A twistor spinor is a spinor field $\varphi \in \Gamma(S)$ satisfying the differential equation

$$\nabla_X \varphi + \frac{1}{n} X \cdot D\varphi = 0$$

for all vector fields X.

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In [18] A. Lichnerowicz introduced the twistor spinors as zeroes of the conformally invariant twistor operator $\mathcal{D}$ and started their systematical geometrical investigation. In particular, using the solution of the Yamabe problem he proved that on compact manifolds the space of all twistor spinors coincides – up to a conformal change of the Riemannian metric – with the space of all Killing spinors (see [19]). Killing spinors are special solutions of the twistor equation. If (M^n, g) admits a non-trivial Killing spinor, then (M^n, g) is an Einstein manifold. In Proposition 2 of this paper we give a generalization of this result for twistor spinors. It is well known that Killing spinors have no zeroes. Whereas non-trivial Killing spinors vanish nowhere, twistor spinors with zeroes are possible. In [12] Th. Friedrich studied the zeroes of twistor spinors. In particular, it turned out that the set of all zeroes of a twistor spinor is a discrete subset of M^n.

We will consider here twistor spinors admitting zeroes on manifolds with parallel Ricci-tensor. As a consequence we will see that any twistor spinor on the standard sphere S^n vanishes at most at one point. Recall that A. Lichnerowicz in [19] proved that the standard sphere S^n is – up to a conformal change of the Riemannian metric – the only compact manifold admitting twistor spinors vanishing at some point.

In [12] Th. Friedrich has shown that twistor spinors on complete connected Einstein manifolds of non-positive scalar curvature vanish at most at one point. We will prove that this result is true also in the case of positive scalar curvature. Furthermore, we will see that the Euclidean space, the hyperbolic space and the sphere are the only complete connected Einstein manifolds admitting twistor spinors vanishing at some point.

Next we consider the conformal deformation of the metric defined by a twistor spinor and give a sufficient condition for completeness of the metric on the complement of the set of all zeroes. Examples are given for which the situation is realized. Finally, we obtain results concerning the conformal vector field defined by a twistor spinor and give a local description of Riemannian manifolds admitting twistor spinors vanishing at some point.

1. General formulas and results

Let (M^n, g) be an n-dimensional Riemannian spin manifold, $n \geq 3$, and let S be the spinor bundle of (M^n, g) equipped with the standard hermitian inner product $\langle\ ,\ \rangle$. The covariant derivative on the spinor bundle induced by the Levi-Cività connection ∇ on M^n will also be denoted by ∇. We consider the Clifford multiplication $\mu : TM^n \otimes S \to S$ and denote by $X \cdot \varphi = \mu(X \cdot \varphi)$ the Clifford multiplication of the vector X by the spinor φ. Let $e_1, \ldots, e_n$ be a local orthonormal frame on M^n. Then the mapping $p : TM^n \otimes S \to TM^n \otimes S$ defined by

$$p(X \otimes \varphi) = X \otimes \varphi + \frac{1}{n} \sum_{j=1}^{n} e_j \otimes e_j \cdot X \cdot \varphi,$$

for $X \otimes \varphi \in TM^n \otimes S$, is a projection of $TM^n \otimes S$ onto the kernel $\ker \mu$ of the Clifford multiplication. The twistor operator $\mathcal{D}$ is defined as the composition of the covariant derivative ∇ and the projection p

$$\mathcal{D} = p \circ \nabla : \Gamma(S) \to \Gamma(T^*M^n \otimes S) \cong \Gamma(TM^n \otimes S) \to \Gamma(\ker \mu).$$

Here we identify the bundles T^*M^n and TM^n via the metric g. Locally the twistor operator $\mathcal{D}$ is given by

$$\mathcal{D}\varphi = \sum_{j=1}^{n} e_j \otimes (\nabla_{e_j}\varphi + \frac{1}{n} e_j \cdot D\varphi) \quad \text{for } \varphi \in \Gamma(S),$$

where D denotes the Dirac operator.

Let $\overline{g} = \lambda g$ be a conformal change of the metric, where λ is a positive real-valued function on M^n, and let $\overline{} : S \to \overline{S}$ denote the natural isomorphism of the corresponding spin bundles. Then we have the relation

$$\overline{\mathcal{D}\varphi} = \lambda^{1/4} \overline{\mathcal{D}}(\lambda^{1/4} \overline{\varphi}) \quad \text{for } \varphi \in \Gamma(S),$$

where $\overline{\mathcal{D}}$ denotes the twistor operator in $\overline{S}$. Thus, the twistor operator $\mathcal{D}$ is a conformally invariant operator.

The zeroes of $\mathcal{D}$ are called twistor spinors. The kernel of $\mathcal{D}$ is described by the twistor equation

$$\nabla_X \varphi + \frac{1}{n} X \cdot D\varphi = 0$$

for all vector fields $X \in \Gamma(TM^n)$. On the space $\ker\mathcal{D}$ of all twistor spinors we have two conformal invariants

$$C_\varphi = \mathrm{Re}\langle D\varphi, \varphi \rangle,$$

$$Q_\varphi = |\varphi|^2 |D\varphi|^2 - C_\varphi^2 - \sum_{j=1}^{n} (\mathrm{Re}\langle D\varphi, e_j \cdot \varphi \rangle)^2 \geq 0,$$

where $e_1, \ldots, e_n$ is an orthonormal frame on M^n. Further, if $\varphi \in \ker\mathcal{D}$ then

$$\nabla_X (D\varphi) = \tfrac{1}{2} n L(X) \cdot \varphi$$

for all $X \in \Gamma(TM^n)$, where L denotes the $(1,1)$-tensor defined by

$$L = \frac{1}{n-2} \left(\frac{R}{2(n-1)} \mathrm{id} - \mathrm{Ric} \right).$$

Furthermore, any twistor spinor satisfies

$$D^2 \varphi = \frac{Rn}{4(n-1)} \varphi.$$

In the case that the manifold (M^n, g) is an Einstein manifold we have some more information. It is easy to prove assuming (M^n, g) to be Einstein that $D(\ker \mathcal{D}) \subset \ker \mathcal{D}$. Moreover, the $(1,1)$-tensor L simplifies to

$$L = \frac{R}{2n(n-1)}\mathrm{id}.$$

We denote by $N_\varphi = \{x \in M^n : \varphi(x) = 0\}$ the zero set of the twistor spinor φ. Th. Friedrich proved (see [12]) that N_φ is a discrete subset of M^n. If φ is a twistor spinor vanishing at some point, then $C_\varphi = 0$ and $Q_\varphi = 0$. Moreover, if φ is a twistor spinor such that $C_\varphi = 0$ and $Q_\varphi = 0$, then we have the relation

$$uD\varphi = \tfrac{1}{2}n \operatorname{grad} u \cdot \varphi,$$

where $u = |\varphi|^2$ (see [12]). Using this relation one proves

Lemma 1.1. *Let (M^n, g) be a Riemannian spin manifold with twistor spinor φ such that $C_\varphi = 0$ and $Q_\varphi = 0$. Then $\varphi/|\varphi|^n$ is a harmonic spinor on $M^n \setminus N_\varphi$.* □

2. A generalization of the Einstein condition

An important special class of twistor spinors are the Killing spinors, i.e. spinor fields $\varphi \in \Gamma(S)$ on (M^n, g) satisfying the differential equation

$$\nabla_X \varphi = \lambda X \cdot \varphi$$

for a complex number λ and all vector fields X. However, a necessary condition for the existence of non-trivial Killing spinors is that the underlying Riemannian manifold is an Einstein manifold of scalar curvature $R = \lambda^2 4n(n-1)$. As a kind of generalization of this condition we are now going to prove the following result.

Proposition 2.1. *Let (M^n, g) be a Riemannian spin manifold, φ a non-trivial twistor spinor on M^n and $u = |\varphi|^2$. Then*

$$u\left\{\frac{R}{n}X - \operatorname{Ric}(X)\right\} = (n-2)\left\{\nabla_X \operatorname{grad} u + \frac{1}{n}\Delta u X\right\}$$

holds for all vector fields X on M^n.

Proof. Let $e_1, \ldots, e_n$ be a local orthonormal frame on M^n. Then we have

$$e_j(u) = -\frac{1}{n}\{\langle e_j \cdot D\varphi, \varphi\rangle + \langle \varphi, e_j \cdot D\varphi\rangle\}$$

for $j = 1, \ldots, n$. This implies

$$X(e_j(u)) = \nabla_X e_j(u) + g(L(X), e_j)u + \frac{2}{n^2}g(X, e_j)|D\varphi|^2.$$

Consequently

$$\nabla_X \operatorname{grad} u = L(X)u + \frac{2}{n^2}|D\varphi|^2 X + \sum_{j=1}^{n}\{\nabla_X e_j(u)e_j + e_j(u)\nabla_X e_j\}.$$

Using $g(\nabla_X e_j, e_i) = -g(e_j, \nabla_X e_i)$ for $i, j = 1, \dots, n$, we obtain

$$\sum_{j=1}^{n}\{\nabla_X e_j(u)e_j + e_j(u)\nabla_X e_j\} = 0.$$

Thus,

$$\nabla_X \operatorname{grad} u = uL(X) + \frac{2}{n^2}|D\varphi|^2 X$$

for all vector fields X. Since

$$|D\varphi|^2 = \frac{Rn}{4(n-1)}u - \frac{n}{2}\Delta u$$

for each twistor spinor φ we conclude that

$$uL(X) = \nabla_X \operatorname{grad} u - \frac{R}{2n(n-1)}uX + \frac{1}{n}\Delta u X.$$

Applying the definition of L, we obtain the assertion. □

Now we obtain the Einstein condition for Killing spinors in the following way. The Killing equation

$$\nabla_X \varphi = \lambda X \cdot \varphi$$

and the identity

$$D^2\varphi = \frac{Rn}{4(n-1)}\varphi$$

for twistor spinors imply the equation

$$\lambda^2 = \frac{R}{4n(n-1)}$$

for the scalar curvature R. Hence λ is either real or imaginary. If φ is a real Killing spinor, then $u = |\varphi|^2$ is constant and by the Proposition (M^n, g) is an Einstein manifold. If φ is an imaginary Killing spinor with Killing number $\lambda = \mathrm{i}\alpha$, then it holds that

$$\nabla_X \operatorname{grad} u = 4\alpha^2 uX$$

for all vector fields X and

$$\Delta u = -4n\alpha^2 u.$$

Thus Proposition 2.1 yields the Einstein condition again.

3. Zeroes of twistor spinors on manifolds with parallel Ricci-tensor

Let (M^n, g) be a Riemannian spin manifold with $\nabla \mathrm{Ric} = 0$. For instance, Einstein manifolds are manifolds with parallel Ricci-tensor. $\nabla \mathrm{Ric} = 0$ implies that the scalar curvature is constant and, moreover, the $(1,1)$-tensor L is parallel, too.

Now consider a geodesic $\gamma(t)$ in M^n and a non-parallel twistor spinor φ. Set $u(t) = |\varphi(\gamma(t))|^2$ and $v(t) = |D\varphi(\gamma(t))|^2$. Moreover, we introduce the functions

$$s(t) = |L(\dot\gamma(t))|,$$
$$A(t) = g(L(\dot\gamma(t)), \dot\gamma(t)).$$

Since $\nabla \mathrm{Ric} = 0$, these functions are constant along the geodesic, $s(t) \equiv s$ and $A(t) \equiv A$. From the twistor equation we obtain the system

$$u'' = Au + \frac{2}{n^2}v, \qquad v'' = \frac{n^2}{2}s^2 u + Av$$

of ordinary differential equations for the functions u and v along $\gamma(t)$.

The Cauchy–Schwarz inequality implies $|A| \le s$, and we consider the following cases

(i) $s = 0, A = 0$,
(ii) $s > 0, A + s = 0$,
(iii) $s > 0, A - s = 0$,
(iv) $s > 0, A + s > 0, A - s < 0$.

Solving the differential equations above, we obtain:

(i) If $s = 0$, then

$$u(t) = \frac{v_0}{n^2}t^2 + u_1 t + u_0, \qquad v(t) = v_0 > 0.$$

(ii) If $s > 0$ and $A + s = 0$, then

$$u(t) = \frac{1}{2}\left\{ \frac{1}{\sqrt{2s}}\left(u_1 - \frac{2}{sn^2}v_1\right)\sin(\sqrt{2s}t) + \left(u_0 - \frac{2}{sn^2}v_0\right)\cos(\sqrt{2s}t) \right.$$
$$\left. + \left(u_1 + \frac{2}{sn^2}v_1\right)t + u_0 + \frac{2}{sn^2}v_0 \right\}$$

and

$$v(t) = \frac{1}{2}\left\{ \frac{1}{\sqrt{2s}}\left(v_1 - \frac{sn^2}{2}u_1\right)\sin(\sqrt{2s}t) + \left(v_0 - \frac{sn^2}{2}u_0\right)\cos(\sqrt{2s}t) \right.$$
$$\left. + \left(v_1 + \frac{sn^2}{2}u_1\right)t + v_0 + \frac{sn^2}{2}u_0 \right\}.$$

(iii) If $s > 0$ and $A - s = 0$, then

$$u(t) = \frac{1}{2}\left\{ \frac{1}{\sqrt{2s}}\left(u_1 + \frac{2}{sn^2}v_1\right)\sinh(\sqrt{2s}t) + \left(u_0 + \frac{2}{sn^2}v_0\right)\cosh(\sqrt{2s}t) + \left(u_1 - \frac{2}{sn^2}v_1\right)t + u_0 - \frac{2}{sn^2}v_0 \right\}$$

and

$$v(t) = \frac{1}{2}\left\{ \frac{1}{\sqrt{2s}}\left(v_1 + \frac{sn^2}{2}u_1\right)\sinh(\sqrt{2s}t) + \left(v_0 + \frac{sn^2}{2}u_0\right)\cosh(\sqrt{2s}t) + \left(v_1 - \frac{sn^2}{2}u_1\right)t + v_0 - \frac{sn^2}{2}u_0 \right\}.$$

(iv) If $s > 0, A + s > 0$ and $A - s < 0$, then

$$u(t) = \frac{1}{2}\left\{ \frac{1}{\sqrt{s+A}}\left(u_1 + \frac{2}{sn^2}v_1\right)\sinh(\sqrt{s+A}t) + \left(u_0 + \frac{2}{sn^2}v_0\right)\cosh(\sqrt{s+A}t) + \frac{1}{\sqrt{s-A}}\left(u_1 - \frac{2}{sn^2}v_1\right)\sin(\sqrt{s-A}t) + \left(u_0 - \frac{2}{sn^2}v_0\right)\cos(\sqrt{s-A}t) \right\}$$

and

$$v(t) = \frac{1}{2}\left\{ \frac{1}{\sqrt{s+A}}\left(v_1 + \frac{sn^2}{2}u_1\right)\sinh(\sqrt{s+A}t) + \left(v_0 + \frac{sn^2}{2}u_0\right)\cosh(\sqrt{s+A}t) + \frac{1}{\sqrt{s-A}}\left(v_1 - \frac{sn^2}{2}u_1\right)\sin(\sqrt{s-A}t) + \left(v_0 - \frac{sn^2}{2}u_0\right)\cos(\sqrt{s-A}t) \right\}.$$

Finally we have

Lemma 3.1. *Let $x_0 \in M^n$ be a zero of the twistor spinor φ, $x_0 \in N_\varphi$. Moreover, let $\gamma(t)$ be a geodesic in M^n such that $\gamma(0) = x_0$. With the notations above we obtain*

(i) If $s = 0$, then

$$u(t) = \frac{v_0}{n^2}t^2 \quad \text{and} \quad v(t) \equiv v_0 > 0.$$

There are no more zeroes of φ along $\gamma(t)$.

(ii) If $s > 0$ and $A + s = 0$, then

$$u(t) = \frac{v_0}{sn^2}(1 - \cos(\sqrt{2s}t)), \qquad v(t) = \frac{v_0}{2}(1 + \cos(\sqrt{2s}t)).$$

Because of the periodicity of u other zeroes along $\gamma(t)$ are possible.

(iii) In the case $s > 0$ and $A - s = 0$ the solutions are

$$u(t) = \frac{v_0}{sn^2}(\cosh(\sqrt{2s}t) - 1), \qquad v(t) = \frac{v_0}{2}(\cosh(\sqrt{2s}t) + 1).$$

No more zeroes are possible.

(iv) For $s > 0, A + s > 0$ and $A - s < 0$ we have

$$u(t) = \frac{v_0}{sn^2}(\cosh(\sqrt{A + s}t) - \cos(\sqrt{s - A}t)),$$

$$v(t) = \frac{v_0}{2}(\cosh(\sqrt{A + s}t) + \cos(\sqrt{s - A}t)).$$

In this case, too, there are no more zeroes. □

Example 3.2. The standard sphere S^n. We have Ric $= (n-1)$id. Thus $L = -\frac{1}{2}$id and $R = n(n-1)$. We fix $x_0 \in S^n$ and a geodesic $\gamma(t)$ of maximum length with $\gamma(0) = x_0$. The constants A and s are given by

$$s = \tfrac{1}{2} \quad \text{and} \quad A + s = 0.$$

Now let φ be a twistor spinor on S^n with zero point x_0. Then we have

$$u(t) = \frac{2v_0}{n^2}(1 - \cos(t)), \qquad v(t) = \frac{v_0}{2}(1 + \cos(t))$$

for the functions $u(t) = |\varphi(\gamma(t))|^2$ and $v(t) = |D\varphi(\gamma(t))|^2$ along $\gamma(t)$. Obviously, $\gamma(2\pi)$ is the next zero of φ along $\gamma(t)$. On the other hand, all maximum geodesics on S^n are closed and of length 2π, i.e. $\gamma(2\pi) = \gamma(0) = x_0$.

Proposition 3.3. *Any twistor spinor on the standard sphere S^n vanishes at most at one point.* □

4. Zeroes of twistor spinors on Einstein manifolds

In [12] Th. Friedrich has shown that twistor spinors on complete connected Einstein manifolds of non-positive scalar curvature vanish at most at one point. We will prove that this result is true also in the case of positive scalar curvature.

Proposition 4.1. *Let (M^n, g) be a complete connected Einstein manifold of positive scalar curvature $R > 0$ and spin. Then any twistor spinor has at most one zero.*

Proof. Since the scalar curvature R of (M^n, g) is positive, we know that

$$s = \frac{R}{2n(n-1)} \quad \text{and} \quad A + s = 0.$$

Let φ be a twistor spinor with two zeroes x_0 and x_1 and let $\gamma(t)$ be a minimizing geodesic such that $\gamma(0) = x_0$ and $\gamma(t_1) = x_1$. Along the geodesic $\gamma(t)$ we have

$$|\varphi(\gamma(t))|^2 = \frac{v_0}{sn^2}(1 - \cos(\sqrt{2s}t)),$$

where $v_0 = |D\varphi(x_0)|^2 > 0$. This implies

$$t_1 = \frac{2k\pi}{\sqrt{2s}} = 2k\frac{\pi}{\sqrt{R/(n(n-1))}} \quad \text{for some positive integer } k.$$

On the other hand, the Theorem of Myers says

$$t_1 \leq \frac{\pi}{\sqrt{R/(n(n-1))}}$$

Thus $k = 0$, and we obtain $x_0 = x_1$. □

Corollary 4.2. *Any twistor spinor on a complete connected Einstein manifold vanishes at most at one point.* □

Remark 4.3. On the Euclidean space $\mathbb{R}^n$, on the hyperbolic space H^n and on the sphere S^n there exist twistor spinors vanishing at some point (see [6]).

5. Twistor spinors on Einstein manifolds

In this section we will see that the Euclidean space, the hyperbolic space and the sphere are the only complete connected Einstein manifolds admitting twistor spinors vanishing at some point. For completeness we recall two results proved in [14].

5.1. Twistor spinors as sums of Killing spinors

Proposition 5.1. *Let (M^n, g) be a Riemannian spin manifold. Suppose that (M^n, g) is an Einstein manifold with non-vanishing scalar curvature $R \neq 0$. Then*

(i) If $R > 0$, then any twistor spinor is the sum of two real Killing spinors.

(ii) If $R < 0$, then any twistor spinor is the sum of two imaginary Killing spinors. □

5.2. A characterization of the hyperbolic space and the Euclidean space

Proposition 5.2. *Let (M^n, g) be a complete connected spin manifold. Furthermore, let (M^n, g) be an Einstein manifold with non-positive scalar curvature $R \leq 0$. Suppose that φ is a non-parallel twistor spinor on M^n such that the function $f : M^n \to [0, \infty)$ defined by $f(x) = |\varphi(x)|^2, x \in M^n$, attains a minimum. Then*

(i) If $R < 0$, then (M^n, g) is isometric to the hyperbolic space with sectional curvature $R/(n(n-1))$.

(ii) If $R = 0$, then (M^n, g) is isometric to the space $\mathbb{R}^n$ with the standard metric. □

Finally we find another characterization of the sphere.

5.3. Characterization of the sphere

Proposition 5.3. *Let (M^n, g) be a complete connected Riemannian spin manifold and an Einstein manifold of positive scalar curvature $R > 0$. Suppose that φ is a twistor spinor on M^n, where $|\varphi|^2$ is non-constant. Then (M^n, g) is isometric to the sphere of scalar curvature R.*

Proof. As a consequence of Proposition 5.2 there exist Killing spinors φ_1 and φ_2 such that $\nabla_X \varphi_1 = \lambda X \cdot \varphi_1$ and $\nabla_X \varphi_2 = -\lambda X \cdot \varphi_2$ for all vector fields X, where $\lambda = \frac{1}{2}\sqrt{R/(n(n-1))}$ and $\varphi = \varphi_1 + \varphi_2$. Thus $|\varphi|^2 = |\varphi_1|^2 + |\varphi_2|^2 + 2(\varphi_1, \varphi_2)$. Now $|\varphi_1|^2$ and $|\varphi_2|^2$ are constant. Consequently, $f = (\varphi_1, \varphi_2)$ is a non-constant function, which satisfies the equation

$$\Delta f = \frac{R}{n-1} f.$$

By the theorem of Myers M^n is compact, and the assertion is a consequence of the theorem of Obata (see [8]). □

6. The conformal deformation defined by a twistor spinor

Let (M^n, g) be a Riemannian spin manifold and suppose that φ is a non-trivial twistor spinor. In earlier papers the metric $\overline{g} = g/|\varphi|^4$ on $M^n \setminus N_\varphi$, where N_φ is the zero set of φ, was studied. For instance, it was shown that $(M^n \setminus N_\varphi, \overline{g})$ is an Einstein manifold (see [12] and [14]). Our purpose here is to prove a sufficient condition for the completeness of the metric $\overline{g}$ on $M^n \setminus N_\varphi$. Recall that N_φ is a discrete subset of M^n.

Let (M^n, g) be a Riemannian spin manifold and suppose that φ is a non-trivial twistor spinor vanishing at some point. We define a function $r = r_\varphi : M^n \to [0, \infty)$ by $r(x) = \mathrm{dist}(x, N_\varphi)$ for $x \in M^n$.

Lemma 6.1. *If $u = |\varphi|^2$ satisfies the condition $u(x) < r(x)$ for all $x \in M^n \setminus N_\varphi$, then $\overline{g}$ is a complete Riemannian metric on $M^n \setminus N_\varphi$.*

Proof. The proof proceeds as the proof of Theorem 1 in [25]. □

Let $\alpha \neq 0$ be a real constant. Since $N_{\alpha\varphi} = N_\varphi$, we have $r_\varphi = r_{\alpha\varphi}$. For the twistor spinor $\alpha\varphi$ the condition above is $\alpha^2|\varphi|^2 < r_{\alpha\varphi}$ which is equivalent to

$$\frac{|\varphi|^2}{r} < \frac{1}{\alpha^2}.$$

Completeness of the Riemannian metric $g/|\varphi|^4$ is equivalent to completeness of the metric $g/|\alpha\varphi|^4$. We deduce:

Proposition 6.2. *Let (M^n, g) be a Riemannian spin manifold and suppose that φ is a non-trivial twistor spinor vanishing at some point. If*

$$\sup_{x \in M^n \setminus N_\varphi} \frac{|\varphi(x)|^2}{\mathrm{dist}(x, N_\varphi)} < \infty,$$

then $\overline{g} = g/|\varphi|^4$ is a complete Riemannian metric on $M^n \setminus N_\varphi$. □

Now we get the following corollary of Proposition 6.2.

Corollary 6.3. *Let (M^n, g) be a complete Riemannian spin manifold and suppose that φ is a non-trivial twistor spinor with a finite number of zeroes and such that the function $|\varphi|^2$ is bounded. Then the metric $\overline{g} = g/|\varphi|^4$ on $M^n \setminus N_\varphi$ is a complete Riemannian metric.*

Proof. When $|\varphi|^2$ is bounded it is enough to prove the condition of Proposition 6.2 in a neighbourhood of the zeroes of φ. Let x_0 be a point of N_φ and $\gamma(t)$ a geodesic with $\gamma(0) = x_0$. Along this geodesic we know that

$$\frac{|\varphi(\gamma(t))|^2}{\mathrm{dist}(\gamma(t), N_\varphi)} = \frac{u(t)}{t}.$$

We have $u'(0) = 0$, so

$$\lim_{t \to 0} \frac{u(t)}{t} = 0.$$

Thus

$$\sup_{x \in M^n \setminus N_\varphi} \frac{|\varphi(x)|^2}{\mathrm{dist}(x, N_\varphi)} < \infty,$$

and the assumption follows. □

Example 6.4. The standard sphere S^n. Let φ be a non-trivial twistor spinor on S^n with zero x_0. Thus x_0 is the only zero of φ. Since S^n is compact, the function $|\varphi|^2$ is bounded. Using Corollary 6.3, the metric $g/|\varphi|^4$ on $S^n \setminus \{x_0\}$ is complete.

7. The conformal vector field defined by a twistor spinor

Let (M^n, g) be a Riemannian spin manifold. We assume φ to be a spinor field on M^n and define a vector field as well as a real-valued 1-form ω by

$$g(V, X) = \omega(X) = \mathrm{i}\langle \varphi, X \cdot \varphi \rangle$$

for all vector fields X.

Lemma 7.1. *Let (M^n, g) be a Riemannian spin manifold and suppose φ to be a non-trivial twistor spinor on M^n. The vector field V has the following properties:*
(i) The Lie derivative of the metric in direction of the vector field V satisfies

$$\mathcal{L}_V g = 2hg, \quad \textit{where } h = \frac{2}{n}\mathrm{Im}(\langle D\varphi, \varphi \rangle);$$

in particular V is a conformal vector field.
(ii) If $C_\varphi = 0$ and $Q_\varphi = 0$, then $||V||^2 = k_\varphi |\varphi|^4$, where $0 \le k_\varphi \le 1$ is a constant. Consequently, if V does not vanish identically, then the zeroes of V coincide with the zeroes of φ. In particular, they are isolated.
(iii) If $C_\varphi = 0$ and $Q_\varphi = 0$, then $k_\varphi = 1$ if and only if $V \cdot \varphi = \mathrm{i}|\varphi|^2 \varphi$.
(iv) Finally, if $C_\varphi = 0$ and $Q_\varphi = 0$, then $V(|\varphi|^2) = h|\varphi|^2$.

Proof. (i) First observe that

$$\begin{aligned} g(\nabla_X V, Y) &= X(g(V, Y)) - g(V, \nabla_X Y) \\ &= \mathrm{i}\{\langle \nabla_X \varphi, Y \cdot \varphi \rangle + \langle \varphi, \nabla_X Y \cdot \varphi + Y \cdot \nabla_X \varphi \rangle - \langle \varphi, \nabla_X Y \cdot \varphi \rangle\} \\ &= -\frac{\mathrm{i}}{n}\{\langle X \cdot D\varphi, Y \cdot \varphi \rangle + \langle \varphi, Y \cdot X \cdot D\varphi \rangle\} \\ &= \frac{2}{n}\mathrm{Im}(\langle X \cdot Y \cdot \varphi, D\varphi \rangle) \end{aligned}$$

for vector fields X and Y on M^n, and hence

$$\begin{aligned} (\mathcal{L}_V g)(X, Y) &= g(\nabla_X V, Y) + g(X, \nabla_Y V) \\ &= \frac{2}{n}\mathrm{Im}(\langle (X \cdot Y + Y \cdot X) \cdot \varphi, D\varphi \rangle) \\ &= \frac{4}{n}\mathrm{Im}(\langle D\varphi, \varphi \rangle) g(X, Y). \end{aligned}$$

(ii) Let $V_\varphi(x)$ be the n-dimensional real subspace of S_x given by

$$V_\varphi(x) = \{X \cdot \varphi(x) : X \in T_x M^n\},$$

where $x \in M^n \setminus N_\varphi$. Then we have

$$\begin{aligned}\mathrm{dist}^2(\mathrm{i}\varphi, V_\varphi) &= |\varphi|^2 - \frac{\sum_{j=1}^n (\mathrm{Re}\langle \mathrm{i}\varphi, e_j \cdot \varphi\rangle)^2}{|\varphi|^2} \\ &= |\varphi|^2 - \frac{\sum_{j=1}^n g(V, e_j)^2}{|\varphi|^2} = |\varphi|^2 - \frac{\|V\|^2}{|\varphi|^2}.\end{aligned}$$

It is well known (cf. [12]) that if $C_\varphi = 0$ and $Q_\varphi = 0$, then the expression

$$\frac{\mathrm{dist}^2(\mathrm{i}\varphi, V_\varphi)}{|\varphi|^2} = 1 - \frac{\|V\|^2}{|\varphi|^4} = c$$

is a constant satisfying $0 \leq c \leq 1$, i.e. $\|V\|^2 = (1-c)|\varphi|^4$.

(iii) This is proved in [28].

(iv) From $C_\varphi = 0$ and $Q_\varphi = 0$ we know that $uD\varphi = \frac{1}{2} n \,\mathrm{grad}\, u \cdot \varphi$, where $u = |\varphi|^2$. Thus

$$V(u) = \mathrm{i}\langle \varphi, \mathrm{grad}\, u \cdot \varphi\rangle = \frac{2}{n}\mathrm{i}u\langle \varphi, D\varphi\rangle = -\frac{2}{n}\mathrm{i}\langle D\varphi, \varphi\rangle u = hu. \qquad \square$$

Proposition 7.2. *Let (M^n, g) be a Riemannian spin manifold and suppose φ to be a non-trivial twistor spinor vanishing at some point x_0. Furthermore, let V be the conformal vector field defined by φ. If $(x_1, \ldots, x_n)$ are local coordinates on M^n near x_0, then*

$$\left(\frac{\partial V^i}{\partial x_j}(x_0)\right)_{i,j=1,\ldots,n} = 0.$$

Proof. Choose a local orthonormal frame $e_1, \ldots, e_n$ and local coordinates $(x_1, \ldots, x_n)$ such that

$$\frac{\partial}{\partial x_j}(x_0) = e_j(x_0)$$

for $j = 1, \ldots, n$. Since

$$V = \sum_{j=1}^n g(V, e_j) e_j = \sum_{m=1}^n V^m \frac{\partial}{\partial x_m},$$

we have

$$g(V, e_j) = \sum_m V^m g(\partial/\partial x_m, e_j)$$

and consequently

$$\frac{\partial}{\partial x_k} g(V, e_j) = \sum_m \left\{ \frac{\partial V^m}{\partial x_m} g(\partial/\partial x_m, e_j) + V^m g(\nabla_{\partial/\partial x_k} \partial/\partial x_m, e_j) \right.$$
$$\left. + V^m g(\partial/\partial x_m, \nabla_{\partial/\partial x_k} e_j) \right\}.$$

Because of $V^m(x_0) = 0$ for $m = 1, \ldots, n$, we have

$$\begin{aligned} \frac{\partial}{\partial x_k} g(V, e_j)(x_0) &= \sum_m \frac{\partial V^m}{\partial x_k}(x_0) g((\partial/\partial x_m)(x_0), e_j(x_0)) \\ &= \sum_m \frac{\partial V^m}{\partial x_k}(x_0)\delta_{mj} \\ &= \frac{\partial V^j}{\partial x_k}(x_0). \end{aligned}$$

On the other hand, we see

$$\begin{aligned} \frac{\partial}{\partial x_k} g(V, e_j) &= \mathrm{i} \frac{\partial}{\partial x_k} \langle \varphi, e_j \cdot \varphi \rangle \\ &= \mathrm{i} \left\{ \langle \nabla_{\partial/\partial x_k} \varphi, e_j \cdot \varphi \rangle + \langle \varphi, \nabla_{\partial/\partial x_k}(e_j \cdot \varphi) \rangle \right\}. \end{aligned}$$

But x_0 is a zero of φ. Hence

$$\frac{\partial}{\partial x_k} g(V, e_j)(x_0) = 0.$$

Consequently

$$\frac{\partial V^j}{\partial x_k}(x_0) = 0$$

for $j, k = 1, \ldots, n$. □

Proposition 7.3. *Let (M^n, g) be a complete connected Riemannian spin manifold and suppose φ to be a non-trivial twistor spinor with a zero x_0. Furthermore, we consider the conformal vector field V and the corresponding function h. Then the following conditions are equivalent:*

(i) h is constant.
(ii) $h \equiv 0$.
(iii) $V \equiv 0$.

Proof. If h is constant, then $\varphi(x_0) = 0$ implies $h \equiv 0$. Thus $\mathcal{L}_V g = 0$, i.e. the 1-parameter transformation group $\{\phi_t : M^n \to M^n\}_{t \in \mathbb{R}}$ defined by V consists of isometries. By Proposition 7.2 we conclude $\mathrm{d}\phi_t(x_0) = \mathrm{id}_{T_{x_0}M^n}$ for each ϕ_t of this family. Let γ be a geodesic with initial point x_0 and initial tangent vector $\dot{\gamma}(x_0)$. Hence $\phi_t(\gamma)$ is also a geodesic with initial point x_0. Since $\mathrm{d}\phi_t(x_0)(\dot{\gamma}(x_0)) = \dot{\gamma}(x_0)$, both geodesics coincide, $\phi_t(\gamma) = \gamma$. Thus $V \equiv 0$ is proved. □

Proposition 7.4. *Let (M^n, g) be a Riemannian spin manifold and assume φ to be a non-parallel twistor spinor vanishing at some point. Furthermore, let h be non-constant. Then the Pfaffian system $\omega = 0$ on $M^n \setminus N_\varphi$ defines an integrable distribution.*

Proof. Let X and Y be vector fields on $M^n \setminus N_\varphi$. Then

$$\begin{aligned} d\omega(X,Y) &= X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \\ &= -\frac{i}{n}\{\langle X\cdot D\varphi, Y\cdot\varphi\rangle + \langle\varphi, Y\cdot X\cdot D\varphi\rangle \\ &\quad -\langle Y\cdot D\varphi, X\cdot\varphi\rangle - \langle\varphi, X\cdot Y\cdot D\varphi\rangle\}. \end{aligned}$$

Since φ has a zero, we can substitute $uD\varphi = \frac{1}{2}n\,\mathrm{grad}\,u\cdot\varphi$ into the equation, and obtain that $u\,d\omega(X,Y) = 2\{X(u)\omega(Y) - Y(u)\omega(X)\}$. Consequently, $d\omega\wedge\omega = 0$ on $M^n\setminus N_\varphi$, and by the Frobenius' Theorem the assertion is proved. □

8. Twistor spinors on non-compact Riemannian manifolds

8.1. The spinor derivative on submanifolds of codimension one

For details we refer to [3] or [5].

Let (M^n, g) be a Riemannian manifold and denote by S the spinor bundle of (M^n, g). Furthermore, let F^{n-1} be an oriented submanifold of codimension one of (M^n, g) with the induced metric. We denote by ξ the normal unit vector field on F^{n-1} given by the orientation of F^{n-1} and M^n. The vector field ξ induces a spin structure on F^{n-1}. We denote by S_F the corresponding spinor bundle of F^{n-1}. Then one can prove the following Lemma (cf. [3]).

Lemma 8.1. *(i) If $n = 2m+1$, then the restriction of the spinor bundle S of (M^n, g) to the submanifold F^{n-1} is isomorphic to the bundle S_F, where ξ acts on $S_F = S_F^+ \oplus S_F^-$ by*

$$\xi\cdot(\varphi^+\oplus\varphi^-) = (-1)^m i\varphi^+ - (-1)^m i\varphi^-$$

and the spinor derivative of $\varphi\in\Gamma(S)$ is given by

$$\nabla_X\varphi = \nabla_X^F(\varphi|_F) - \tfrac{1}{2}\nabla_X\xi\cdot\xi\cdot\varphi$$

for all $X\in T_xF^{n-1}$.

(ii) If $n = 2m+2$, then the restriction of the spinor bundle S of (M^n, g) to the submanifold F^{n-1} is isomorphic to the bundle $S_F\oplus\widehat{S_F}$, where ξ acts on $S_F\oplus\widehat{S_F}$ by

$$\xi\cdot(\varphi_1\oplus\widehat{\varphi_2}) = (-1)^m i(\varphi_2\oplus\widehat{\varphi_1})$$

and the spinor derivative of $\varphi \in \Gamma(S)$ is given by

$$\nabla_X \varphi = \nabla^F_X \varphi_1 \oplus \widehat{\nabla}^F_X \widehat{\varphi_2} - \tfrac{1}{2} \nabla_X \xi \cdot \xi \cdot \varphi$$

for all $X \in T_x F^{n-1}$ (with the denotation $\varphi|_F \cong \varphi_1 \oplus \widehat{\varphi_2}$). Here $\widehat{\ }: S \to \widehat{S}$ is the isomorphism of the vector bundles given in [3]. □

Moreover, we will need the following Lemma (cf. [3]).

Lemma 8.2. *Let (M^n, g) be an odd-dimensional spin manifold. Then the isomorphism $\widehat{\ }: S \to \widehat{S}$ of the vector bundles has the properties*

$$\widehat{\nabla^S_X \varphi} = \nabla^{\widehat{S}}_X \widehat{\varphi}, \qquad \widehat{X \cdot \varphi} = -X \cdot \widehat{\varphi}$$

for all vector fields X and spinor fields φ. □

8.2. A local description of Riemannian manifolds admitting twistor spinors vanishing at some point

Proposition 8.3. *Let (M^n, g) be a Riemannian manifold and suppose φ to be a twistor spinor such that $N_\varphi \neq \emptyset$ and h is non-constant. Then $(M^n \setminus N_\varphi, g)$ is locally isometric to a warped product*

$$\left(F^{n-1} \times (-\varepsilon, \varepsilon), \exp\left(2 \int_0^t h(\phi_s(x))\, \mathrm{d}s \right) g_F \oplus \mathrm{d}t^2 \right),$$

where (F^{n-1}, g_F) is a Riemannian spin manifold. Moreover, if we have $k_\varphi = 1$, then the restriction $\varphi|_F \in \Gamma(S_F)$ of φ yields a non-trivial twistor spinor on (F^{n-1}, g_F).

Proof. Let V be the conformal vector field defined by φ. Then we consider on $M^n \backslash N_\varphi$ the unit vector field $\xi = V/|\varphi|^2$. We fix an arbitrary point $x \in M^n \backslash N_\varphi$. Denote by $\phi_t(x)$ the integral curves of V with initial point x and let F^{n-1} be a leaf of the foliation of $M^n \setminus N_\varphi$ defined by V (see Proposition 7.4) with $x \in F^{n-1}$. We compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} (\phi_t^* g)_x (X, Y) &= (\mathcal{L}_V g)_{\phi_t(x)} (\mathrm{d}\phi_t(X), \mathrm{d}\phi_t(Y)) \\ &= 2h(\phi_t(x))\, g_{\phi_t(x)} (\mathrm{d}\phi_t(X), \mathrm{d}\phi_t(Y)) \\ &= 2h(\phi_t(x))\, (\phi_t^* g)_x (X, Y)\end{aligned}$$

for $X, Y \in T_x F^{n-1}$. The solution of this differential equation is given by

$$(\phi_t^* g)_x (X, Y) = g_x(X, Y) \exp\left(2 \int_0^t h(\phi_s(x))\, \mathrm{d}s \right).$$

Let $F^{n-1}\times(-\varepsilon,\varepsilon)$ be a neighbourhood of F^{n-1} and define $\Phi : F^{n-1}\times(-\varepsilon,\varepsilon)\to M^n\setminus N_\varphi$ by $\Phi(x,t)=\phi_t(x)$. Thus

$$\mathrm{d}\Phi_{(x,t)}(X\oplus r\,\partial/\partial t)=\mathrm{d}\phi_t(X)+r\xi(\phi_t(x)),$$

which implies

$$\begin{aligned}
&(\Phi^*g)_{(x,t)}(X\oplus r\,\partial/\partial t, Y\oplus p\,\partial/\partial t)\\
&\quad= g_{\phi_t(x)}(\mathrm{d}\phi_t(X)\oplus r\xi(\phi_t(x)),\mathrm{d}\phi_t(Y)\oplus p\xi(\phi_t(x)))\\
&\quad= (\phi_t^*g)_x(X,Y)+rp\\
&\quad= \exp\left(2\int_0^t h(\phi_s(x))\,\mathrm{d}s\right)g_x(X,Y)+\mathrm{d}t^2(r\,\partial/\partial t,p\,\partial/\partial t).
\end{aligned}$$

Therefore

$$\Phi^*(g)=\exp\left(2\int_0^t h(\phi_s(x))\,\mathrm{d}s\right)g|_F\oplus\mathrm{d}t^2.$$

If $k_\varphi=1$, then $\xi\cdot\varphi=\mathrm{i}\varphi$.

Let $n=2m+1$. From

$$\nabla_X\varphi=\nabla_X^F(\varphi|_F)-\tfrac{1}{2}\nabla_X\xi\cdot\xi\cdot\varphi,$$

we obtain

$$\begin{aligned}
\nabla_X\varphi&=\nabla_X^F(\varphi|_F)-\tfrac{1}{2}\mathrm{i}(\mathrm{i}\nabla_X\varphi-\xi\cdot\nabla_X\varphi)\\
&=\nabla_X^F(\varphi|_F)+\tfrac{1}{2}\nabla_X\varphi+\tfrac{1}{2}\mathrm{i}\xi\cdot\nabla_X\varphi.
\end{aligned}$$

Since φ has a zero, we can substitute $uD\varphi=\frac{1}{2}n\,\mathrm{grad}\,u\cdot\varphi$ into the equation above. Thus

$$\begin{aligned}
2u\nabla_X^F(\varphi|_F)&=\tfrac{1}{2}\{\mathrm{i}\xi\cdot X\cdot\mathrm{grad}\,u\cdot\varphi-X\cdot\mathrm{grad}\,u\cdot\varphi\}\\
&=\tfrac{1}{2}\{-\mathrm{i}X\cdot\xi\cdot\mathrm{grad}\,u\cdot\varphi-X\cdot\mathrm{grad}\,u\cdot\varphi\}\\
&=\mathrm{i}\xi(u)X\cdot\varphi-X\cdot\mathrm{grad}\,u\cdot\varphi\\
&=X\cdot\{h\xi-\mathrm{grad}\,u\}\cdot\varphi,
\end{aligned}$$

i.e.

$$\nabla_X^F(\varphi|_F)=\frac{1}{2u}X\cdot\{h\xi-\mathrm{grad}\,u\}\cdot\varphi.$$

Let $X_1,\dots,X_{n-1}$ be a local orthonormal frame tangential to F^{n-1}. Then for the Dirac operator D^F of F^{n-1} we have

$$
\begin{aligned}
D^F(\varphi|_F) &= \sum_{j=1}^{n-1} X_j \cdot \nabla^F_{X_j}(\varphi|_F) \\
&= \frac{1}{2u}\sum_{j=1}^{n-1} X_j^2 \cdot \{h\xi - \operatorname{grad} u\} \cdot \varphi \\
&= -\frac{n-1}{2u}\{h\xi - \operatorname{grad} u\} \cdot \varphi .
\end{aligned}
$$

Obviously $\varphi|_F$ is a solution of the twistor equation on F^{n-1}.

Now let $n = 2m + 2$. From $\xi \cdot \varphi = \mathrm{i}\varphi$ we derive

$$
\begin{aligned}
\xi \cdot \varphi|_F &= \xi \cdot (\varphi_1 \oplus \widehat{\varphi_2}) \\
&= (-1)^m \mathrm{i}(\varphi_2 \oplus \widehat{\varphi_1}) \\
&= \mathrm{i}\varphi|_F \;=\; \mathrm{i}\varphi_1 \oplus \widehat{\varphi_2}.
\end{aligned}
$$

Hence $\varphi_1 = (-1)^m \varphi_2$ and $\varphi|_F = \varphi_1 \oplus (-1)^m \widehat{\varphi_1}$ respectively. Using Lemma 8.1, we have

$$
\nabla_X \varphi = \nabla^F_X \varphi_1 \oplus (-1)^m \widehat{\nabla}^F_X \widehat{\varphi_1} - \tfrac{1}{2}\nabla_X \xi \cdot \xi \cdot \varphi
$$

for all X tangential to F^{n-1}. Analogously to the first case we have

$$
\nabla^F_X \varphi_1 \oplus (-1)^m \widehat{\nabla}^F_X \widehat{\varphi_1} = \frac{1}{2u} X \cdot \{h\xi - \operatorname{grad} u\} \cdot \varphi .
$$

This implies

$$
\begin{aligned}
D^F\varphi_1 \oplus (-1)^{m+1}\widehat{D^F\varphi_1} &= \sum_{j=1}^{n-1} X_j \cdot \nabla^F_{X_j}\varphi_1 \oplus (-1)^{m+1}\widehat{\left(X_j \cdot \nabla^F_{X_j}\varphi_1\right)} \\
&= \sum_{j=1}^{n-1} X_j \cdot \nabla^F_{X_j}\varphi_1 \oplus (-1)^m X_j \cdot \widehat{\nabla}^F_{X_j}\widehat{\varphi_1} \\
&= -\frac{n-1}{2u}\{h\xi - \operatorname{grad} u\} \cdot \varphi .
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
\nabla^F_X \varphi_1 \oplus (-1)^m \widehat{\nabla^F_X \varphi_1} &= -\frac{1}{n-1} X \cdot \{D^F\varphi_1 \oplus (-1)^{m+1}\widehat{D^F\varphi_1}\} \\
&= -\frac{1}{n-1}\{X \cdot D^F\varphi_1 \oplus (-1)^m \widehat{\left(X \cdot D^F\varphi_1\right)}\}.
\end{aligned}
$$

Consequently, we arrive at

$$
\nabla^F_X \varphi_1 = -\frac{1}{n-1} X \cdot D^F \varphi_1 ,
$$

i.e. φ_1 is a non-trivial twistor spinor on F^{n-1}. □

8.3. The spinor derivative on warped products

For details see again [3] or [5].

Let (F^{n-1}, g_F) be a spin manifold with spin structure (Q_F, f_F), $I = (a,b) \subset \mathbb{R}$, $-\infty \le a < b \le \infty$, be an open interval and $c \in C^\infty(F \times I, \mathbb{R})$ be a smooth function. We consider the warped product

$$(M^n, g) := (F \times I, \mathrm{e}^{2c(x,t)} g_F \oplus \mathrm{d}t^2).$$

(M^n, g) is a spin manifold with spin structure (Q, f). This spin structure reduces with respect to the global unit vector field $\xi(x,t) := (\partial/\partial t)(x)$ to a spin structure $(\widehat{Q}, \widehat{f})$, which is on each fibre $F \times \{t\}$ the conformally equivalent spin structure of $(F, \mathrm{e}^{2c(x,t)} g_F)$ to (Q_F, f_F). Denote by S_F the spinor bundle of (F^{n-1}, g_F). Let $\pi : F \times I \to F$ be the projection. For a section $\varphi \in \Gamma(\pi^* S_F)$ we denote by $\varphi_t \in \Gamma(S_F)$ the spinor field, given by $\varphi_t(x) = \varphi(x,t)$. For a vector field X on F let $\widetilde{X}$ be the vector field defined by $\widetilde{X}(x,t) := \mathrm{e}^{-c(x,t)} X(x)$ on M.

Lemma 8.4. *On the warped product (M^n, g) we have the following relations between the spinor bundles of (M^n, g) and (F^{n-1}, g_F).*

(i) If $n = 2m + 1$, then the spinor bundle S of (M^n, g) can be identified with the bundle $\pi^* S_F$

$$\pi^* S_F \xrightarrow{\sim} S, \quad \varphi \to \widetilde{\varphi}$$

in such a way that the Clifford multiplication satisfies

$$\begin{aligned} &\widetilde{X}(x,t) \cdot \widetilde{\varphi}(x,t) = \widetilde{(X(x) \cdot \varphi_t(x))}, \quad X(x) \in T_x F, \\ &\xi \cdot \widetilde{(\varphi^+ \oplus \varphi^-)} = (-1)^m \mathrm{i} \widetilde{(\varphi^+ - \varphi^-)} \end{aligned}$$

and the spinor derivative is given by

$$\begin{aligned} &\nabla^S_{\widetilde{X}} \widetilde{\varphi} = \mathrm{e}^{-c} \widetilde{\left(\nabla^{S_F}_X \varphi_t - X \cdot \mathrm{grad}_F(c) \cdot \varphi_t\right)} - \tfrac{1}{2} c' \widetilde{X} \cdot \xi \cdot \widetilde{\varphi}, \quad X \in T_x F, \\ &\nabla^S_\xi \widetilde{\varphi} = \widetilde{(\partial/\partial t)(\varphi)}. \end{aligned}$$

(ii) If $n = 2m + 2$, then the spinor bundle S of (M^n, g) can be identified with the bundle $\pi^* S_F \oplus \pi^* \widehat{S_F}$

$$\begin{aligned} &\pi^* S_F \oplus \pi^* \widehat{S_F} \xrightarrow{\sim} S \\ &\varphi = \varphi_1 \oplus \widehat{\varphi_2} \to \widetilde{\varphi} = \widetilde{\varphi_1} \oplus \widetilde{\widehat{\varphi_2}} \end{aligned}$$

in such a way that the Clifford multiplication satisfies

$$\widetilde{X}(x,t)\cdot\widetilde{\varphi}(x,t) = (X(x)\cdot\varphi_{1t}(x))\widetilde{\ }\oplus (X(x)\cdot\varphi_{2t}(x))\widetilde{\ }, \quad X\in T_xF,$$
$$\xi\cdot(\widetilde{\varphi_1}\oplus\widetilde{\widehat{\varphi_2}}) = (-1)^m \mathrm{i}(\widetilde{\varphi_2}\oplus\widetilde{\widehat{\varphi_1}})$$

and the spinor derivative is given by

$$\nabla^S_{\widetilde{X}}\widetilde{\varphi} = \mathrm{e}^{-c}\left\{\left(\nabla^{S_F}_X\varphi_{1t} - X\cdot\mathrm{grad}_F(c)\cdot\varphi_{1t}\right)\widetilde{\ } \oplus \left(\nabla^{\widehat{S_F}}_X\widehat{\varphi_{2t}} - X\cdot\mathrm{grad}_F(c)\cdot\widehat{\varphi_{2t}}\right)\widetilde{\ }\right\} - \tfrac{1}{2}c'\widetilde{X}\cdot\xi\cdot\widetilde{\varphi}, \quad X\in T_xF,$$

$$\nabla^S_\xi\widetilde{\varphi} = \widetilde{(\partial/\partial t)}(\varphi),$$

where $c' = (\partial/\partial t)(c)$*. Furthermore,* $\mathrm{grad}_F(c)$ *is defined by* $\mathrm{grad}(c) = \widetilde{\mathrm{grad}_F}(c) \oplus c'\xi$ *and* $\mathrm{grad}(c)$ *is the gradient on* M^n *respectively* g*. For the definition of* $\widehat{S_F}$ *we refer again to [3] or [5].*

Proof. Let $(e_1,\dots,e_{n-1})$ be a local orthonormal frame of (F,g_F) and let $X\in T_xF$. On the warped product (M^n,g) the Levi-Cività connection satisfies

$$g(\nabla^M_{\widetilde{X}}\widetilde{e_j},\widetilde{e_i}) = \mathrm{e}^{-c}\left\{g_F(\nabla^F_X e_j,e_i) + e_j(c)g_F(X,e_i) - e_i(c)g_F(X,e_j)\right\},$$
$$g(\nabla^M_{\widetilde{X}}\widetilde{e_j},\xi) = -(\partial/\partial t)(c)g_F(X,e_j),$$
$$g(\nabla^M_\xi\widetilde{e_j},\xi) = g(\nabla^M_\xi\widetilde{e_j},\widetilde{e_i}) = 0 \quad \text{for } i,j = 1,\dots,n-1.$$

Now the assertion follows similarly as in the proof of Lemma 4 in [5]. □

8.4. The existence of twistor spinors on warped products

Let (F,g_F) be a n-dimensional Riemannian spin manifold. Denote by S_F the spinor bundle and by ∇ the covariant derivative on S_F induced by the Levi-Cività connection on F. Furthermore, let $c\in C^\infty(F,\mathbb{R})$ be a smooth function on F. We define a covariant derivative

$$\nabla^c : \Gamma(S_F)\to\Gamma(T^*F\otimes S_F) \quad \text{by}$$
$$\nabla^c_X\psi = \nabla_X\psi - X\cdot\mathrm{grad}(c)\cdot\psi$$

for all vector fields X and spinor fields ψ.

We call a section $\psi\in\Gamma(S_F)$ c-parallel if and only if $\nabla^c\psi = 0$.

Lemma 8.5. *Let* $\psi\in\Gamma(S_F)$ *be a non-trivial c-parallel spinor field on* (F,g_F)*. Then* ψ *is a twistor spinor and has the following properties:*
(i) $C_\psi = 0$ *and* $Q_\psi = 0$*.*
(ii) $|\psi|^2\cdot\mathrm{e}^{2c} = constant$*.*

(iii) ψ has no zero.

Proof. From $\nabla_X \psi = X \cdot \operatorname{grad}(c) \cdot \psi$ for all vector fields X it follows that $D\psi = -n \operatorname{grad}(c) \cdot \psi$. Thus $C_\psi = \operatorname{Re}(\langle D\psi, \psi\rangle) = 0$ and

$$\begin{aligned} Q_\psi &= |D\psi|^2|\psi|^2 - \sum_j \operatorname{Re}(\langle e_j \cdot D\psi, \psi\rangle)^2 \\ &= n^2 \|\operatorname{grad}(c)\|^2 |\psi|^4 - n^2 \|\operatorname{grad}(c)\|^2 |\psi|^4 \\ &= 0. \end{aligned}$$

Since ψ is a twistor spinor, it holds that

$$|\psi|^2 D\psi = \tfrac{1}{2} n \operatorname{grad} |\psi|^2 = -n|\psi|^2 \operatorname{grad}(c) \cdot \psi.$$

Then

$$\operatorname{grad}(\ln |\psi|^2) = -2 \operatorname{grad}(c).$$

Therefore (ii) follows. Now (iii) is obvious. □

Example 8.6. Let (F, g) be a Riemannian spin manifold admitting a non-trivial parallel spinor field ψ and suppose c to be an arbitrary real-valued function on F. Then $e^{-c}\overline{\psi}$ on $(F, \overline{g})$, where $\overline{g} = e^{-4c} g$, is a c-parallel spinor field. Here $\overline{} : S_F \to \overline{S}_F$ denotes the natural isomorphism of the corresponding spin bundles.

Remark 8.7. There are a lot of manifolds admitting parallel spinor fields, which hence yield c-parallel spinor fields as mentioned above.

Corollary 8.8. *$\psi \in \Gamma(S_F)$ is a non-trivial c-parallel spinor field on (F, g_F) if and only if $\overline{\psi}/|\psi|$ is a non-trivial parallel spinor field on $(F, \overline{g_F} = e^{4c} g_F)$.*

Proof. The assertion is a direct consequence of Lemma 8.5, the example of this section and Proposition 6 in [12]. □

Proposition 8.9. *Let (F^{n-1}, g_F) be a Riemannian spin manifold, $c \in C^\infty(F \times \mathbb{R}, \mathbb{R})$ be a smooth function and suppose $\psi \in \Gamma(S_F)$ to be a non-trivial c-parallel spinor field on F. Then there exists a non-trivial twistor spinor $\widetilde{\varphi}$ on the warped product*

$$(M^n, g) := (F^{n-1} \times \mathbb{R}, e^{2c} g_F \oplus dt^2)$$

such that $C_{\widetilde{\varphi}} = 0$ and $Q_{\widetilde{\varphi}} = 0$.

Proof. Let $n = 2m + 1$. We define a spinor field $\widetilde{\varphi}$ on (M^n, g) by

$$\varphi(x, t) = e^{c(x,t)/2} \psi(x).$$

Then

$$\nabla_{\widetilde{X}}\widetilde{\varphi} = \mathrm{e}^{-c}\,\widetilde{(\nabla_X\psi - X\cdot \mathrm{grad}_F(c)\cdot\psi)} - \tfrac{1}{2}c'\widetilde{X}\cdot\xi\cdot\widetilde{\varphi}$$
$$= -\tfrac{1}{2}c'\widetilde{X}\cdot\xi\cdot\widetilde{\varphi} \quad \text{for } X\in\Gamma(TF),$$
$$\nabla_\xi\widetilde{\varphi} = \widetilde{(\partial/\partial t)}(\varphi) = \tfrac{1}{2}c'\widetilde{\varphi}.$$

From this we conclude

$$D\widetilde{\varphi} = \sum_{j=1}^{2m}\widetilde{e}_j\cdot\nabla_{\widetilde{e}_j}\widetilde{\varphi} + \xi\cdot\nabla_\xi\widetilde{\varphi} = \tfrac{1}{2}(2m+1)c'\xi\cdot\widetilde{\varphi},$$

where $(e_1,\ldots,e_{2m})$ is a local orthonormal frame on (F, g_F). Thus

$$\nabla_{\widetilde{X}}\widetilde{\varphi} + \frac{1}{2m+1}\widetilde{X}\cdot D\widetilde{\varphi} = -\tfrac{1}{2}c'\widetilde{X}\cdot\xi\widetilde{\varphi} + \tfrac{1}{2}c'\widetilde{X}\cdot\xi\cdot\widetilde{\varphi} = 0$$

and

$$\nabla_\xi\widetilde{\varphi} + \frac{1}{2m+1}\xi\cdot D\widetilde{\varphi} = \tfrac{1}{2}c'\widetilde{\varphi} - \tfrac{1}{2}c'\widetilde{\varphi} = 0.$$

Consequently, $\widetilde{\varphi}$ is a twistor spinor on (M^n, g).

Let $n = 2m+2$. We consider the spinor field $\widetilde{\varphi}$ on (M^n, g) given by

$$\varphi(x,t) = \mathrm{e}^{c(x,t)/2}(\psi(x)\oplus(-1)^m\widehat{\psi}(x)).$$

We obtain

$$\nabla_{\widetilde{X}}\widetilde{\varphi} = \mathrm{e}^{-c}\left\{\widetilde{\left(\nabla_X\psi - X\cdot\mathrm{grad}_F(c)\cdot\psi\right)}\oplus\widetilde{\left(\widehat{\nabla}_X\widehat{\psi} - X\cdot\mathrm{grad}_F(c)\cdot\widehat{\psi}\right)}\right\}$$
$$-\tfrac{1}{2}c'\widetilde{X}\cdot\xi\cdot\widetilde{\varphi}$$
$$= -\tfrac{1}{2}c'\widetilde{X}\cdot\xi\cdot\widetilde{\varphi} \quad \text{for } X\in\Gamma(TF)\ ,$$
$$\nabla_\xi\widetilde{\varphi} = \widetilde{(\partial/\partial t)}(\varphi) = \tfrac{1}{2}c'\widetilde{\varphi}.$$

Analogously, $\widetilde{\varphi}$ is a twistor spinor on (M^n, g). We have

$$\langle D\widetilde{\varphi}, \widetilde{\varphi}\rangle = \tfrac{1}{2}nc'\langle\xi\cdot\widetilde{\varphi}, \widetilde{\varphi}\rangle,$$

and thus

$$C_{\widetilde{\varphi}} = \mathrm{Re}(\langle D\widetilde{\varphi}, \widetilde{\varphi}\rangle) = 0$$

follows. Furthermore

$$
\begin{aligned}
Q_{\widetilde{\varphi}} &= |\widetilde{\varphi}|^2 |D\widetilde{\varphi}|^2 - \sum_{j=1}^{n-1} (\mathrm{Re}\langle D\widetilde{\varphi}, \widetilde{e_j} \cdot \widetilde{\varphi}\rangle)^2 - (\mathrm{Re}\langle D\widetilde{\varphi}, \xi \cdot \widetilde{\varphi}\rangle)^2 \\
&= |\widetilde{\varphi}|^2 |D\widetilde{\varphi}|^2 - \tfrac{1}{4} n^2 (c')^2 \sum_{j=1}^{n-1} (\mathrm{Re}\langle \xi \cdot \widetilde{\varphi}, \widetilde{e_j} \cdot \widetilde{\varphi}\rangle)^2 - (\mathrm{Re}\langle D\widetilde{\varphi}, \xi \cdot \widetilde{\varphi}\rangle)^2 \\
&= \tfrac{1}{4} n^2 (c')^2 |\widetilde{\varphi}|^4 - \tfrac{1}{4} n^2 (c')^2 (\mathrm{Re}\langle \xi \cdot \widetilde{\varphi}, \xi \cdot \widetilde{\varphi}\rangle)^2 \\
&= 0. \qquad \square
\end{aligned}
$$

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